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Heisenberg realization for $U_q(sl_n)$ on the flag manifold

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Abstract

We give the Heisenberg realization for the quantum algebra $U_q(sl_n)$, which is written by the q -difference operator on the flag manifold. We construct it from the action of $U_q(sl_n)$ on the q -symmetric algebra $\mathcal{A}_q(Mat_n)$ by the Borel-Weil like approach. Our realization is applicable to the construction of the free field realization for the $U_q(\widehat{sl_n})$ [AOS].

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1. Introduction

Recently, the quantum Knizhnik-Zamolodchikov equations (q -KZ eq.) [Sm, FR] have been analyzed [M1, R]. This q -KZ equations are important both for physics and mathematics by the relationship with 2-dimensional integrable theories [Sm, DFJMN], quantum affine Lie algebras and elliptic R -matrices [FR, DJO].

To solve the classical ($q = 1$) KZ equations, an important and powerful tools were the free field realization for the affine Lie algebra $\widehat{\mathcal{G}}$ [W, FF] and the Heisenberg realization for the corresponding Lie algebra \mathcal{G} which is written by the differential operator on the flag manifold [SV, ATY, FM]. Even in the quantum case ($q \neq 1$), for example for the algebra $U_q(\widehat{sl_2})$, the Heisenberg realization and the free field realization [FJ, M2, ABG, Sh] are also important for the analysis of the q -KZ equation [JMMN, KQS, M3]. We expect that this situation is the same for other quantum affine Lie algebras.

The aim of this paper is to construct the Heisenberg realization for the quantum algebra $U_q(sl_n)$. In the forthcoming paper [AOS], the free field realization for the quantum affine algebra $U_q(\widehat{sl_n})$ will be constructed by using this Heisenberg realization.

2. Quantum algebra $U_q(sl_n)$

§ 2.1. First we fix some notations. The algebra $U_q(sl_n)$ is generated by e_i, f_i and invertible k_i ($1 \leq i \leq n-1$) with relations

$$\begin{aligned} k_i e_j k_i^{-1} &= q^{A_{ij}} e_j, & \sum_{m=0}^{1-A_{ij}} (-1)^m \begin{bmatrix} 1-A_{ij} \\ m \end{bmatrix} e_i^{1-A_{ij}-m} e_j e_i^m &= 0, \\ k_i f_j k_i^{-1} &= q^{-A_{ij}} f_j, & \sum_{m=0}^{1-A_{ij}} (-1)^m \begin{bmatrix} 1-A_{ij} \\ m \end{bmatrix} f_i^{1-A_{ij}-m} f_j f_i^m &= 0, \\ e_i f_j - f_j e_i &= \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}}, \end{aligned}$$

where $q \in \mathbf{C}$, $(A_{ij})_{1 \leq i, j \leq n-1}$ is the Cartan matrix such that $A_{ij} = 2\delta_{ij} - \delta_{i, j+1} - \delta_{i, j-1}$, $\begin{bmatrix} n \\ m \end{bmatrix} = [n]!/[n-m]![m]!$ and $[n] = (q^n - q^{-n})/(q - q^{-1})$.

The algebra $U_q(sl_n)$ is a Hopf algebra with the comultiplication Δ

$$\Delta(k_i) = k_i \otimes k_i, \quad \Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i,$$

the antipode S such that $S(k_i) = k_i^{-1}$, $S(e_i) = -k_i^{-1}e_i$, $S(f_i) = -f_i k_i$ and the co-unit ϵ such that $\epsilon(k_i) = 1$, $\epsilon(e_i) = 0$, $\epsilon(f_i) = 0$.

§ **2.2.** Let M_λ be the Verma module over $U_q(sl_n)$ generated by the highest weight vector $|\lambda\rangle$ such that $e_i|\lambda\rangle = 0$, $k_i|\lambda\rangle = q^{\lambda_i}|\lambda\rangle$ with $\lambda_i \in \mathbf{C}$. The dual module M_λ^* is generated by $\langle\lambda|$ which satisfies $\langle\lambda|f_i = 0$, $\langle\lambda|k_i = q^{\lambda_i}\langle\lambda|$. The bilinear form $M_\lambda^* \otimes M_\lambda \rightarrow \mathbf{C}$ is uniquely defined by $\langle\lambda|\lambda\rangle = 1$ and $(\langle u|X)|v\rangle = \langle u|(X|v\rangle)$ for any $\langle u| \in M_\lambda^*$, $|v\rangle \in M_\lambda$ and $X \in U_q(sl_n)$.

3. Heisenberg realization for $U_q(sl_n)$

§ **3.1.** The Heisenberg algebra \mathcal{H}_n is generated by the coordinate x_{ij} , $x_{ij}^{-1} \in \mathbf{C}$ and the differential operator $\vartheta_{ij} = x_{ij} \frac{\partial}{\partial x_{ij}}$ ($1 \leq i < j \leq n$) with relation $[\vartheta_{ij}, x_{kl}] = \delta_k^i \delta_l^j x_{kl}$ or equivalently

$$q^{\vartheta_{ij}} x_{kl} q^{-\vartheta_{ij}} = q^{\delta_k^i \delta_l^j} x_{kl}.$$

The quantum algebra $U_q(sl_n)$ is realized by the Heisenberg algebra \mathcal{H}_n . We have

Theorem I. *There exists the algebra homomorphism $\pi_\lambda : U_q(sl_n) \rightarrow \mathcal{H}_n$ define as*

$$\begin{aligned} \pi_\lambda(k_i) &= q^{\sum_{j=1}^{i-1} (\vartheta_{ji} - \vartheta_{j \ i+1}) + (\lambda_i - 2\vartheta_{i \ i+1}) + \sum_{j=i+2}^n (\vartheta_{i+1 \ j} - \vartheta_{ij})}, \\ \pi_\lambda(e_i) &= \sum_{k=1}^i q^{\sum_{j=1}^{k-1} (\vartheta_{ji} - \vartheta_{j \ i+1})} \frac{x_{ki}}{x_{k \ i+1}} [\vartheta_{k \ i+1}], \\ \pi_\lambda(f_i) &= \sum_{k=1}^{i-1} \frac{x_{k \ i+1}}{x_{ki}} [\vartheta_{ki}] q^{-\sum_{j=k+1}^{i-1} (\vartheta_{ji} - \vartheta_{j \ i+1}) - (\lambda_i - 2\vartheta_{i \ i+1}) - \sum_{j=i+2}^n (\vartheta_{i+1 \ j} - \vartheta_{ij})} \\ &\quad + x_{i \ i+1} [(\lambda_i - \vartheta_{i \ i+1}) + \sum_{j=i+2}^n (\vartheta_{i+1 \ j} - \vartheta_{ij})] \\ &\quad - \sum_{k=i+2}^n \frac{x_{ik}}{x_{i+1 \ k}} [\vartheta_{i+1 \ k}] q^{\lambda_i + \sum_{j=k}^n (\vartheta_{i+1 \ j} - \vartheta_{ij})}, \end{aligned}$$

with $x_{ii} = 1$.

Here $[n]$ denotes the q integer, so $\pi_\lambda(g)$'s are the q -difference operators. The proof will be given in the next section.

We also have the following dual generators[†]

[†] These dual generators relate to the screening currents of the free field realization for $U_q(\widehat{sl_n})$ [AOS] which must important to the analysis of the q -KZ equation.

Theorem II. *There exists the algebra anti-homomorphism $\tilde{\pi}_\lambda : U_q(sl_n) \rightarrow \mathcal{H}_n$, $\tilde{\pi}_\lambda = \tilde{\sigma} \circ \pi_\lambda \circ \sigma$, with σ such that $\sigma(k_i) = k_{n-i}$, $\sigma(e_i) = e_{n-i}$, $\sigma(f_i) = f_{n-i}$ and $\tilde{\sigma}$ such that $\tilde{\sigma}(x_{ij}) = x_{n+1-j, n+1-i}$, $\tilde{\sigma}(\vartheta_{ij}) = -\vartheta_{n+1-j, n+1-i}$, $\tilde{\sigma}(\lambda_i) = -\lambda_{n+1-i}$.*

§ 3.2. Let $\mathcal{F} = \mathbf{C}[x_{ij}]|0\rangle$ be the Fock module over Heisenberg algebra \mathcal{H}_n generated by the highest weight vector $|0\rangle$ such that $x_{ij}^{-1}|0\rangle = \vartheta_{ij}|0\rangle = 0$. The dual module $\mathcal{F}^* = \langle 0| \mathbf{C}[x_{ij}^{-1}]$ is generated by $\langle 0|$ which satisfies $\langle 0|x_{ij} = \langle 0|\vartheta_{ij} = 0$. The bilinear form $\mathcal{F}^* \otimes \mathcal{F} \rightarrow \mathbf{C}$ is uniquely defined by $\langle 0|0\rangle = 1$ and $(\langle u|X)|v\rangle = \langle u|(X|v\rangle)$ for any $\langle u| \in \mathcal{F}^*$, $|v\rangle \in \mathcal{F}$ and $X \in \mathcal{H}_n$. For $\langle 0|f(x_{ij}^{-1}) \in \mathcal{F}^*$ and $g(x_{ij})|0\rangle \in \mathcal{F}$, $\langle 0|f(x_{ij}^{-1})g(x_{ij})|0\rangle$ is nothing but the constant part of $f(x_{ij}^{-1})g(x_{ij})$.

4. Construction of the Heisenberg realization for $U_q(sl_n)$

Next we prove above Theorems by a Borel-Weil like approach, which is based on the method in Ref. [N]. First we give some notations.

§ 4.1. The q -symmetric algebra $\mathcal{A}_q(Mat_n)$ is generated by t_{ij} ($1 \leq i, j \leq n$) with relations

$$(4.1) \quad \begin{aligned} t_{ik}t_{jk} &= qt_{jk}t_{ik}, & t_{il}t_{jk} &= t_{jk}t_{il}, \\ t_{ik}t_{il} &= qt_{il}t_{ik}, & t_{ik}t_{jl} - qt_{il}t_{jk} &= t_{jl}t_{ik} - q^{-1}t_{jk}t_{il}, \end{aligned}$$

for $i < j$ and $k < l$. Note that this algebra has the algebra automorphism ρ such that $\rho(t_{ij}) = t_{n+1-j, n+1-i}$, $\rho(q) = q^{-1}$ and the algebra anti-automorphism $\tilde{\rho}$ such that $\tilde{\rho}(t_{ij}) = t_{n+1-j, n+1-i}$, $\tilde{\rho}(q) = q$.

The algebra $\mathcal{A}_q(Mat_n)$ has the structure of a $U_q(sl_n)$ -module. The action of $U_q(sl_n)$ on $\mathcal{A}_q(Mat_n)$ is

$$\begin{aligned} k_m t_{ij} &= t_{ij} q^{\delta_{mj} - \delta_{m+1, j}}, & e_m t_{ij} &= t_{ij-1} \delta_{m+1, j}, & f_m t_{ij} &= t_{ij+1} \delta_{mj}, \\ g(uv) &= \sum_a (g'_a u)(g''_a v), & g \cdot 1 &= \epsilon(g) 1, \end{aligned}$$

for all $u, v \in \mathcal{A}_q(Mat_n)$ and for all $g \in U_q(sl_n)$ with $\Delta(g) = \sum_a g'_a \otimes g''_a$. Note that this action of $g \in U_q(sl_n)$ can be written by the matrix $\varrho(g)_{ij}$ as $g t_{ij} = \sum_k t_{jk} \varrho(g)_{kj}$ with $\varrho(k_m) = q^{E_{mm} - E_{m+1, m+1}}$, $\varrho(e_m) = E_{m, m+1}$, $\varrho(f_m) = E_{m+1, m}$ and $(E_{\alpha\beta})_{ij} = \delta_{\alpha i} \delta_{\beta j}$. These matrices are

noting but the vector representation for the $U_q(sl_n)$. The action for the rows of matrix t_{ij} is given by the above automorphism ρ or $\tilde{\rho}$.

§ 4.2. For the ordered set $I = \{i_1 < \cdots < i_r\}$ and $J = \{j_1 < \cdots < j_r\}$, let ξ_J^I be the quantum r -minor determinant with respect to rows I and columns J such that [TT, NYM]

$$\xi_J^I = \sum_{\sigma \in \mathbf{S}_r} (-q)^{l(\sigma)} t_{i_{\sigma(1)}j_1} \cdots t_{i_{\sigma(r)}j_r}.$$

Here \mathbf{S}_r is the permutation group of the set $\{1, \dots, r\}$ and $l(\sigma)$ stands for the number of inversions involved in σ ; $l(\sigma) = \#\{(i, j); i < j, \sigma(i) > \sigma(j)\}$. From now on, $\xi_J^I = 0$ if I or J has same elements. Note that $\xi_J^I \xi_{J'}^{I'} = \xi_{J'}^{I'} \xi_J^I$ if $I' \subset I, J' \subset J$. We have

Proposition. *With the lower triangular matrix B , the Gauss decomposition of the matrix $T = (t_{ij})$ of the q -coordinates is given as*

$$t_{ij} = \sum_k B_{ik} X_{kj}, \quad B_{ij} = (\xi_1^1 \cdots \xi_{j-1}^{j-1})^{-1} \xi_1^1 \cdots \xi_j^{j-1}, \quad X_{ij} = (\xi_1^1 \cdots \xi_i^i)^{-1} \xi_1^1 \cdots \xi_{i-1}^i,$$

and $B_{ij} = 0$ for $i < j$ and $X_{ij} = 0$ for $i > j$. Here $\{1 \cdots 0\} = \{\}$.

Proof. follows from

$$t_{ij} = B_{i1} X_{1j} + (\xi_1^1)^{-1} \xi_{1j}^1, \quad (\xi_1^1 \cdots \xi_r^r)^{-1} \xi_1^1 \cdots \xi_r^i = B_{i,r+1} X_{r+1,j} + (\xi_1^1 \cdots \xi_{r+1}^{r+1})^{-1} \xi_1^1 \cdots \xi_{r+1}^i,$$

which are obtained from the q -deformed Jacobi identity

$$\xi_1^1 \cdots \xi_r^r \xi_1^1 \cdots \xi_{r+1}^{r+1} \xi_{r+2}^{r+2} = \xi_1^1 \cdots \xi_{r+1}^{r+1} \xi_1^1 \cdots \xi_{r+2}^{r+2} - q \xi_1^1 \cdots \xi_{r+2}^{r+2} \xi_1^1 \cdots \xi_{r+1}^{r+1}. \quad \text{Q.E.D.}$$

We regard X_{ij} ($i < j$) as a q -analogue of local coordinates of the flag manifold $B \backslash GL_n$. For $i < i_1$ and $I = \{i_1 < \cdots < i_r\}$, we denote $\eta_I^i = \xi_1^1 \cdots \xi_{i_1}^{i+1} \cdots \xi_{i_r}^{i+r}$, then $X_{ij} = (\eta_i^{i-1})^{-1} \eta_j^{i-1}$. Since the principal minors $\xi_1^1 \cdots \xi_i^i$'s $1 \leq i \leq n$ commute with each other, one can consistently adjoin their inverse to the algebra $\mathbf{C}[\xi_J^I]$.

§ 4.3. The quantum minor η_{ij}^r 's satisfy, for $r < i < j < k < l$, the same relations as t_{ij} 's in (4.1) and Plücker relation (Young symmetry) [TT, NYM, N]

$$\eta_i^r \eta_{jk}^r - q \eta_j^r \eta_{ik}^r + q^2 \eta_k^r \eta_{ij}^r = 0,$$

and the commutation relations

$$\eta_i^r \eta_{jk}^r = q \eta_{jk}^r \eta_i^r, \quad \eta_{ij}^r \eta_k^r = q \eta_k^r \eta_{ij}^r, \quad \eta_{ik}^r \eta_j^r = \eta_k^r \eta_{ij}^r + \eta_i^r \eta_{jk}^r,$$

$$\eta_{ij}^r \eta_{jk}^r = q \eta_{jk}^r \eta_{ij}^r, \quad \eta_{ij}^r \eta_{kl}^r = q^2 \eta_{kl}^r \eta_{ij}^r.$$

The action of $U_q(sl_n)$ on the quantum minor η_j^i is

$$k_m \eta_{ij}^{i-1} = \eta_{ij}^{i-1} q^{\delta_{mj} - \delta_{m+1,j} + \delta_{mi}},$$

$$e_m \eta_{ij}^{i-1} = \eta_{i,j-1}^{i-1} \delta_{m+1,j}, \quad f_m \eta_{ij}^{i-1} = \eta_{i,j+1}^{i-1} \delta_{mj} + \eta_{i+1,j}^{i-1} \delta_{mi}.$$

Owing to the Plücker relation, $\eta_{i+1,j}^{i-1} = \eta_{i+1}^{i-1} (\eta_i^{i-1})^{-1} \eta_{ij}^{i-1} - q \eta_j^{i-1} (\eta_i^{i-1})^{-1} \eta_{i+1}^{i-1}$, the algebra $\mathcal{A} = \mathbf{C}[\eta_j^{i-1}, (\eta_i^{i-1})^{-1}]_{1 \leq i \leq n-1, i \leq j \leq n}$ has the structure of a $U_q(sl_n)$ -module.

§ 4.4. To relate the non-commutative algebra $\mathbf{C}[X_{ij}]$ with the commutative one $\mathbf{C}[x_{ij}]$, we fix the ordering of η_j^i 's. The algebra \mathcal{A} has the basis

$$\{ (\eta_n^0)^{a_{1n}} \dots (\eta_1^0)^{a_{11}} (\eta_n^1)^{a_{2n}} \dots (\eta_2^1)^{a_{22}} \dots (\eta_{n-1}^{n-2})^{a_{n-1,n-1}} \mid a_{ij} \in \mathbf{Z}_{\geq 0}, i < j, a_{ii} \in \mathbf{Z} \},$$

which ordering we call *normal ordering*. We introduce the projection $\circ * \circ : \mathcal{A} \rightarrow \mathcal{A}$ such that

$$\circ \text{ any ordered } \prod_{i \leq j} (\eta_j^{i-1})^{a_{ij}} \circ = \text{normal ordered } \prod_{i \leq j} (\eta_j^{i-1})^{a_{ij}}.$$

Let $Z_\lambda^a = \circ \prod_i (\eta_i^{i-1})^{\lambda_i} \prod_{j < k} (X_{jk})^{a_{jk}} \circ$ with $\lambda_i \in \mathbf{Z}$ and $a_{ij} \in \mathbf{Z}_{\geq 0}$. If we denote $Y^i = (\eta_n^{i-1})^{a_{in}} \dots (\eta_i^{i-1})^{a_{ii}}$ with $a_{ii} = \lambda_i - \sum_{j=i+1}^n a_{ij}$, then $Z_\lambda^a = Y^1 \dots Y^{n-1}$. The algebra \mathcal{A} has the decomposition $\mathcal{A} = \oplus_{\lambda_i \in \mathbf{Z}} \mathcal{A}_\lambda$ such that \mathcal{A}_λ is the vector space spanned by the vectors $\{Z_\lambda^a \mid a_{ij} \in \mathbf{Z}_{\geq 0}, i < j\}$. The algebra \mathcal{A}_λ also has the structure of a $U_q(sl_n)$ -module, and we have

Lemma. *The left action of $U_q(sl_n)$ on \mathcal{A}_λ is as follows*

$$\begin{aligned} k_i Z_\lambda^a &= Z_\lambda^a q^{\sum_{j=1}^{i-1} (a_{ji} - a_{j,i+1}) + (\lambda_i - 2a_{i,i+1}) + \sum_{j=i+2}^n (a_{i+1,j} - a_{ij})}, \\ e_i Z_\lambda^a &= \sum_{k=1}^i q^{\sum_{j=1}^{k-1} (a_{ji} - a_{j,i+1})} [a_{k,i+1}] \circ Z_\lambda^a (X_{k,i+1})^{-1} X_{ki} \circ, \\ f_i Z_\lambda^a &= \sum_{k=1}^{i-1} [a_{ki}] \circ X_{k,i+1} (X_{ki})^{-1} Z_\lambda^a \circ q^{-\sum_{j=k+1}^{i-1} (a_{ji} - a_{j,i+1}) - (\lambda_i - 2a_{i,i+1}) - \sum_{j=i+2}^n (a_{i+1,j} - a_{ij})} \\ &\quad + [(\lambda_i - a_{i,i+1}) + \sum_{j=i+2}^n (a_{i+1,j} - a_{ij})] \circ X_{i,i+1} Z_\lambda^a \circ \\ &\quad - \sum_{k=i+2}^n [a_{i+1,k}] \circ X_{ik} (X_{i+1,k})^{-1} Z_\lambda^a \circ q^{\lambda_i + \sum_{j=k}^n (a_{i+1,j} - a_{ij})}. \end{aligned}$$

Proof. follows from

$$\begin{aligned}
k_m Y^i &= Y^i q^{(a_{im} - a_{i, m+1})} \sum_{j=i}^{n-1} \delta_{mj} + \delta_{m, i-1} \sum_{j=i+1}^n a_{m+1, j}, \\
e_m Y^i &= [a_{i, m+1}] \circ Y^i (\eta_{m+1}^{i-1})^{-1} \eta_m^{i-1} \circ \sum_{j=i+1}^n \delta_{m+1, j}, \\
f_m Y^i &= [a_{im}] \circ \eta_{m+1}^{i-1} (\eta_m^{i-1})^{-1} Y^i \circ \sum_{j=i}^{n-1} \delta_{mj} + \delta_{m, i-1} \sum_{k=i+1}^n [a_{ik}] \eta_{ik}^{i-2} \circ (\eta_k^{i-1})^{-1} Y^i \circ q^{-\sum_{j=i+1}^{k-1} a_{ij}} \\
&= [a_{im}] \circ \eta_{m+1}^{i-1} (\eta_m^{i-1})^{-1} Y^i \circ \sum_{j=i}^{n-1} \delta_{mj} + \delta_{m, i-1} \left[\sum_{k=i+1}^n a_{ik} \right] \circ \eta_i^{i-2} (\eta_{i-1}^{i-2})^{-1} Y^i \circ \\
&\quad - \delta_{m, i-1} \sum_{k=i+1}^n [a_{ik}] \circ \eta_k^{i-2} (\eta_{i-1}^{i-2})^{-1} \eta_k^{i-1} (\eta_i^{i-1})^{-1} Y^i \circ q^{-\sum_{j=k}^n a_{ij}},
\end{aligned}$$

here we use $k_m(\eta_i^r)^a = (k_m \eta_i^r)^a$, $e_m(\eta_i^r)^a = [a](\eta_i^r)^{a-1} (e_m \eta_i^r)$, $f_m(\eta_i^r)^a = [a](f_m \eta_i^r)(\eta_i^r)^{a-1}$ with $a \in \mathbf{Z}$ and the identity $\sum_k [a_k] q^{(\sum_{j < k} - \sum_{j > k}) a_j} = [\sum_k a_k]$. The polynomials of q in $e_i Z_\lambda^a$ and $f_i Z_\lambda^a$ come from the Cartan parts of the comultiplication of e_i and f_i respectively.

Q.E.D.

§ 4.5. Proof of Theorem I.

We consider the commutative algebra $\mathbf{C}[x_{ij}]_{1 \leq i < j \leq n}$ and define an isomorphism $\pi_\lambda : \mathcal{A}_\lambda \rightarrow \mathbf{C}[x_{ij}]$ by $\pi_\lambda(Z_\lambda^a) = z^a$, with $z^a = \prod_{r < j} (x_{rj})^{a_{rj}}$. Applying this isomorphism π_λ to above Lemma, we obtain the q -difference operators on $\mathbf{C}[x_{ij}]$ in Theorem I.

Q.E.D.

§ 4.6. Proof of Theorem II.

With the lower triangular matrix \tilde{B} , the Gauss decomposition of inverse direction $T = \tilde{X} \tilde{B}$ is obtained by the algebra anti-automorphism $\tilde{\rho}$ in §4.1 from the Gauss decomposition $T = BX$. By the algebra automorphism ρ with some sign changing, we get the action of $U_q(sl_n)$ on $\mathbf{C}[\tilde{X}_{ij}]$ and the dual generators of Theorem II.

Q.E.D.

Conclusion and Discussion.

We constructed the Heisenberg realization for the $U_q(sl_n)$ by the flag coordinate, which is applicable to the construction of the free field realization for the $U_q(\widehat{sl_n})$ [AOS]. In the Ref. [DJMM], they also gave the similar realization for the $U_q(sl_n)$ but it seems that it can not be affinized.

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Appendix. The Jordan-Schwinger type realization and q -oscillator

§ **A.1.** If we consider only $i = 1$ of $t_{ij} \in \mathcal{A}_q(Mat_n)$, then we can obtain the n variables Jordan-Schwinger type realization for $U_q(sl_n)$ [H, Z]. Let us denote $t_j = t_{1j}$, the algebra $\mathcal{A} = \mathbf{C}[t_i]_{1 \leq i \leq n}$ has the basis $\{ t_n^{a_n} \cdots t_1^{a_1} \mid a_i \in \mathbf{Z}_{\geq 0} \}$, which ordering we call *normal ordering*, and has the structure of a $U_q(sl_n)$ -module. By an isomorphism $\pi : \mathcal{A} \rightarrow \mathbf{C}[x_i]$, $\pi(t_n^{a_n} \cdots t_1^{a_1}) = x_1^{a_1} \cdots x_n^{a_n}$ and by the action of $U_q(sl_n)$ on \mathcal{A} , we obtain

Proposition. *There exists the algebra homomorphism $\pi : U_q(sl_n) \rightarrow \mathcal{H}_n$ define as*

$$\pi(k_i) = q^{\vartheta_i - \vartheta_{i+1}}, \quad \pi(e_i) = \frac{x_i}{x_{i+1}}[\vartheta_{i+1}], \quad \pi(f_i) = \frac{x_{i+1}}{x_i}[\vartheta_i].$$

We introduce the projection $\circ * \circ$ same as before. Denote $X_i = t_1^{-1}t_i$ ($2 \leq i \leq n$) and $Z_\lambda^a = \circ t_1^\lambda \prod_{i=2}^n X_i^{a_i} \circ = t_n^{a_n} \cdots t_1^{a_1}$ with $a_1 = \lambda - \sum_{i=2}^n a_i$, then the algebra $\mathcal{A}[t_1^{-1}]$ has the decomposition $\mathcal{A}[t_1^{-1}] = \oplus_{\lambda \in \mathbf{Z}} \mathcal{A}_\lambda$ such that \mathcal{A}_λ is the vector space spanned by the vectors $\{ Z_\lambda^a \mid a_i \in \mathbf{Z}_{\geq 0}, i > 1 \}$. By an isomorphism $\pi_\lambda : \mathcal{A}_\lambda \rightarrow \mathbf{C}[x_i]$, $\pi_\lambda(t_n^{a_n} \cdots t_1^{a_1}) = x_2^{a_2} \cdots x_n^{a_n}$ and by the action of $U_q(sl_n)$ on \mathcal{A}_λ , we obtain the $n - 1$ variables inhomogeneous realization for $U_q(sl_n)$, which is the same as above Proposition with additional conditions $x_1 = 1$ and $\vartheta_1 = \lambda - \sum_{i=2}^n \vartheta_i$. This realization corresponds with that in Theorem-I on $\mathbf{C}[x_{1j}]$ with $\lambda_i = 0$ for $i \neq 1$.

§ **A.2.** For the Heisenberg algebra $\langle x, \vartheta \rangle$ with $q^\vartheta x q^{-\vartheta} = qx$, if we denote

$$a = x, \quad a^\dagger = \frac{1}{x}[\vartheta], \quad N = \vartheta,$$

then $\langle a, a^\dagger, N \rangle$ satisfies the q -oscillator algebra such that

$$aa^\dagger = [N], \quad a^\dagger a = [N + 1],$$

which is equivalent to $a^\dagger a - q^{\pm 1} a a^\dagger = q^{\mp N}$. And they satisfy $[N, a] = a$, $[N, a^\dagger] = -a^\dagger$.

So we can rewrite our Theorem by the q -oscillator algebra.

References

- [ABG] A. Abada, A. Bougourzi and M. El Gradechi, *Deformation of the Wakimoto construction*, (hep-th/9209009) preprint CRM-1829 (1992).
- [AOS] H. Awata, S. Odake and J. Shiraishi, *Free Boson representation of $U_q(\widehat{sl_3})$* , (hep-th/9305017) preprint RIMS-920,YITP/K-1017;*Free Boson realization of $U_q(\widehat{sl_N})$* , (hep-th/9305146) preprint RIMS-924,YITP/K-1018
- [ATY] H. Awata, A. Tsuchiya and Y. Yamada, *Integral formulas for the WZNW correlation functions*, Nucl. Phys. **B365** (1991) 680-696.
H. Awata, *Screening current Ward identity and integral formulas for the WZNW correlation functions*, Prog. Theor. Phys. Supplement **110** (1992) 303-319.
- [DFJMN] B. Davies, O. Foda, M. Jimbo, T. Miwa and A. Nakayashiki, *Diagonalization of the XXZ Hamiltonian by vertex operators*, Commun. Math. Phys. **151** (1993) 89-154.
- [DJMM] E. Date, M. Jimbo, K. Miki and T. Miwa, *Cyclic representations of $U_q(sl(n+1, \mathbf{C}))$ at $q^N = 1$* , Publ. RIMS. Kyoto Univ. **27** (1991) 347-366.
- [DJO] E. Date, M. Jimbo and M. Okado, *Crystal base and q -vertex operators*, Osaka Univ. Math. Sci. preprint **1** (1991).
- [FF] B. Feigin and E. Frenkel, *A family of representations of affine Lie algebras*, Russian Math. Surveys **43** (1988) 221-222; *Affine Kac-Moody algebras and semi-infinite Flag manifolds*, Commun. Math. Phys. **128** (1990) 161-189.
- [FJ] I. Frenkel and N. Jing, *Vertex representations of quantum affine algebras*, Proc. Nat'l. Acad. Sci. USA **85** (1988) 9373-9377.
- [FM] B. Feigin and F. Malikov, *Integral intertwining operators and complex powers of differential (q -difference) operators*, preprint RIMS-894 (1992).
- [FR] I. Frenkel and N. Reshetikhin, *Quantum affine algebras and holonomic difference equations*, Commun. Math. Phys. **146** (1992) 1-60.
- [H] T. Hayashi, *Q -analogues of Clifford and Weyl algebras - Spinor and Oscillator representations of quantum enveloping algebras*, Commun. Math. Phys. **127** (1990) 129-144.
- [JMMN] M. Jimbo, K. Miki, T. Miwa and A. Nakayashiki, *Correlation functions of the XXZ model for $\Delta < -1$* , Phys. Lett. **A168** (1992) 256-263.
- [KQS] A. Kato, Y. Quano and J. Shiraishi, *Free Boson Representation of q -Vertex Operators and their Correlation Functions*, Tokyo Univ. preprint UT-618 (1992).
- [M1] A. Matsuo, *Jackson integrals of Jordan-Pochhammer type and quantum Knizhnik-Zamolodchikov equations*, Nagoya Univ. preprint (1992); *Quantum algebra structure of certain Jackson integrals*, Nagoya Univ. preprint (1992).
- [M2] A. Matsuo, *Free field representation of quantum affine algebra $U_q(\widehat{sl_2})$* , (hep-th/9208079) Nagoya Univ. preprint (1992).

- [M3] A. Matsuo, *Free field representation of q -deformed primary fields for $U_q(\widehat{sl_2})$* , (hep-th/9212040) Nagoya Univ. preprint (1992).
- [N] M. Noumi, *Quantum Grassmannians and q -hypergeometric series*, CWI Quarterly **5** (1992) 293-307.
- [NYM] M. Noumi, H. Yamada and K. Mimachi, *Finite dimensional representations of the quantum group $GL_q(n; \mathbf{C})$ and the zonal spherical functions on $U_q(n-1) \backslash U_q(n)$* , to appear in Japanese J. Math.
- [R] N. Reshetikhin, *Jackson-type integrals, Bethe vectors, and solutions to a difference analog of the Knizhnik-Zamolodchikov system*, Lett. Math. Phys. **26** (1992) 153-165.
- [Sh] J. Shiraishi, *Free boson representation of $U_q(\widehat{sl_2})$* , Phys. Lett. **A171** (1992) 243-248.
- [Sm] F. Smirnov, *Dynamical symmetries of massive integrable models*, Int. J. Mod. Phys. **A7** Supplement 1 (1992) 813-837;839-858.
- [SV] V. Schechtman and A. Varchenko, *Arrangements of hyperplanes and Lie algebra homology*, Invent. Math. **106** (1991) 139-194.
- [TT] E. Taft and J. Towber, *Quantum deformation of flag schemes and Grassmann schemes I, J*, Algebra **142** (1991) 1-36.
- [W] M. Wakimoto, *Fock representations of the affine Lie algebra $A_1^{(1)}$* , Commun. Math. Phys. **104** (1986) 605-609.
- [Z] see references in C. Zachos, *Paradigms of quantum algebras*, Symmetries in Science **V**, B. Gruber et al. (eds.), Plenum, 1991, p.593-609.